**Indian Statistical Institute, Bangalore**
B. Math.(Hons.) I Year, Second Semester
Semestral Examination
Analysis -II

Time: 3 hours May 3, 2010 Instructor: Pl.Muthuramalingam

Maximum Marks 50

1. a) For any matrix $A = ((a_{ij})), i = 1, 2, \cdots, n,\ j = 1, 2, \cdots, k,\ a_{ij}$ real, define $\| A \|$ by $\| A \| = [\sum_{i,j} \mid a_{ij} \mid^2]^{\frac{1}{2}}$. If $A, B$ are matrices such that $AB$ is also a matrix show that

$$\| AB \| \leq \| A \| \| B \| .$$

[2]

b) Let $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. Show that $\|AB\| \neq \| A \| \| B \|$. [1]

c) $A_k, A, B_k, B \epsilon M_{n\times n}(R)-$ the space of $n\times n$ real matrices. If $\| A_k - A \| + \| B_k - B \| \longrightarrow 0$ as $k \longrightarrow \infty$ then show that $\| A_kB_k - AB \| \longrightarrow 0$ as $k \longrightarrow \infty$. [2]

d) Let $G_1, G_2 : M_{n\times n}(R) \longrightarrow M_{n\times n}(R)$ have total derivative at $X_0$. Define $F : M_{n\times n}(R) \longrightarrow M_{n\times n}(R)$ by $F(X) = G_1(X)G_2(X)$. Let the error functions $E_1(X_0, U), E_2(X_0, U), E(X_0, U)$ for $U$ in $M_{n\times n}(R)$ be given by

$$E_1(X_0, U) = G_1(X_0 + U) - G_1(X_0) - G_1'(X_0)U$$
$$E_2(X_0, U) = G_2(X_0 + U) - G_2(X_0) - G_2'(X_0)U$$
$$E(X_0, U) = F(X_0 + U) - F(X_0) - G_1'(X_0)UG_2(X_0) - G_1(X_0)G_2'(X_0)U.$$

Verify that $E(X_0, U) =$

$$E_1(X_0, U)G_2(X_0+U)+G_1(X_0)E_2(X_0, U)+G_1^1(X_0)U[G_2(X_0+U)-G_2(X_0)]$$

or verify that $E(X_0, U) =$

$$G_1(X_0+U)E_2(X_0, U)+E_1(X_0, U)G_2(X_0)+[G_1(X_0+U)-G_1(X_0)]G_2'(X_0)U.$$

[3]

e) Show that $F$ has a total derivative at $X_0$. Find $F'(X_0)U$ in terms of $X_0, U, G_1, G_2, G_1', G_2'$. [3]

2. Let $f : R^2 \longrightarrow R$ be a function such that the derivatives $\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}$ exist and both the derivatives are continuous. Show that $f$ has a total derivative. [4]

3. a) Let $g : R^2 \longrightarrow R$ be given by

$$g(x, y) = \begin{cases} \frac{xy^2}{x^2+y^4} & x \neq 0, \\ 0 \text{ for } x = 0. \end{cases}$$

Show that the directional derivative $g'(\vec{O}, \vec{u})$ exists for each direction $\vec{u}$. at $\vec{O} = (0, 0)$. [2]

b) Show that $g$ is not continuous at $\vec{O}$. [1]

4. If $f : R^n \longrightarrow R$ has total derivative at $\vec{x}_0$, then $f$ is continuous at $\vec{x}_0$. [2]

5. a) Let $f, g : [a, b] \longrightarrow R$ be both bounded and $f$ is Riemann integrable. If $\{x : f(x) \neq g(x)\} = \{x_0\}$ for some $x_0$ in $(a, b)$ show that $g$ is Riemann integrable. [4]

b) Further show that $\int\limits_a^b f = \int\limits_a^b g$ [2]

6. Let $(X, d)$ be a matric space with a countable dense set $D$. If $\mathbf{C} = \{B(y, \frac{1}{r}) : r = 1, 2, 3, 4, \cdots, y \in D\}$, show that every open set can be writen as union of elements form $\mathbf{C}$. [3]

7. a) Let $(X, d)$ be a connected metric space. If $A$ is a nonempty closed and open subset of $X$, than show that $A = X$. [1]

b) Let $G$ be any open connected subset of $R^2$. Show that any two points of $G$ can be joined by a path consisting of line segments parallel to the coordinate axes. [4]

c) Let $G_2$ be an open connected subset of $R^2$ and $0 \in G_2$. If $f : G_2 \longrightarrow R$ satisfies $f(0) = 0$, $\frac{\partial f}{\partial x_1} \equiv 0 \equiv \frac{\partial f}{\partial x_2}$, then show that $f(x) = 0$ for all $x$ in $G_2$. [3]

8. a) In a metric space $(X, d)$ prove: $\mid d(x, y) - d(a, b) \mid \leq d(x, a) + d(y, b)$. [2]

b) Show that $d : X \times X \longrightarrow [0, \infty)$ is a continuous function. Here $X \times X$ is given the metric $m$ :

$$m((x_1, x_2), (y_1, y_2)) = \left\{[d(x_1, y_1)]^2 + [d(x_2, y_2)]^2\right\}^{\frac{1}{2}}.$$

[1]

9. If $J$ is a compact, connected subset of $R$ with at least tow points, then show that $J = [a, b]$ for some $a < b$. [2]

10. Let $N = \{1, 2, 3, \cdots\}$ with the metric $d(x, y) = 1$ if $x \neq y$ and $d(x, x) = 0$. Clearly $N$ is a bounded and closed subset of $(N, d)$. Show that $(N, d)$ is not a compact metric space. [2]

11. a) $\{(x, y) \in R^2 : 0 < x^2 + y^2 \leq 1\}$ is not compact. [1]

    b) $\{(x, y) \in R^2 : xy = 1\}$ is not compact. [1]

    c) Let $f : \cup_{n=1}^{\infty}[n, n + a_n] \longrightarrow R, a_n \geq 0$ and $a_n \longrightarrow 0$ as $n \longrightarrow \infty$, be given by $f(x) = x^2$. If $f$ is uniformly continuous, show that $na_n \longrightarrow 0$. [1]

    d) Let $G = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \text{ real}, \ ad - bc \neq 0 \right\}$. Show that $G$ is an open subset of $M_{2\times 2}(R)$. [1]

    e) Let $G$ be as in $(d)$. Show that $G$ is not connected. [Hint: Find $f : G \longrightarrow Y, f$ continuous, onto, $Y$ disconnected]. [2]